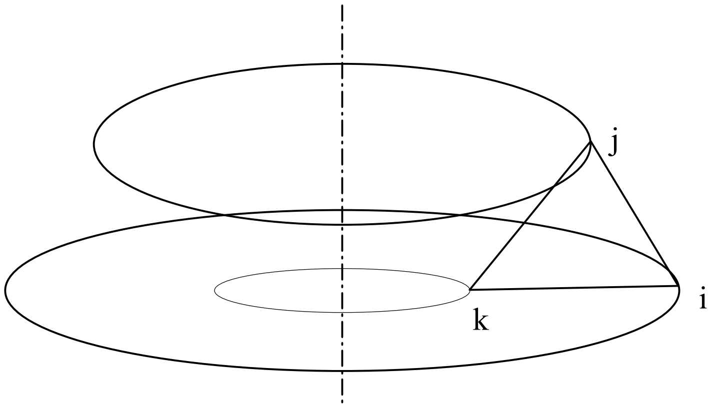
FEM – 2D (x,y) teplotni pole



Dvoudimenzialni teplotni pole (r,z) je popsano rovnici:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)+Q=0 \tag{1}$$

pro kterou plati tyto okrajove podminky:

na S_1 $\quad T=T_B$ (2)

na S_2 $\quad \lambda\frac{\partial T}{\partial r}l_r+\lambda\frac{\partial T}{\partial z}l_z+q+\alpha\left(T-T_\infty\right)=0$ (3)

Q...zdroj tepla
λ...tepelna vodivost
α...soucinitel prestupu tepla proudenim
$l_r,\ l_z$...smerove cosiny

$$\left\{G^{(e)}\right\}=\int_{V^{(e)}}\left[\mathrm{N}^{(e)}\right]^T\left(\frac{\lambda}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\lambda\frac{\partial^2 T}{\partial z^2}+Q\right)dV \tag{4}$$

$$\left[\mathrm{N}^{(e)}\right]^T\lambda\frac{\partial^2 T}{\partial z^2}=\lambda\frac{\partial}{\partial z}\left(\left[\mathrm{N}^{(e)}\right]^T\frac{\partial T}{\partial z}\right)-\lambda\frac{\partial\left[\mathrm{N}^{(e)}\right]^T}{\partial z}\frac{\partial T}{\partial z} \tag{5}$$

$$\left[\mathrm{N}^{(e)}\right]^T\left(\frac{\lambda}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right)=\frac{\lambda}{r}\frac{\partial}{\partial r}\left(\left[\mathrm{N}^{(e)}\right]^T r\frac{\partial T}{\partial r}\right)-\lambda\frac{\partial\left[\mathrm{N}^{(e)}\right]^T}{\partial r}\frac{\partial T}{\partial r} \tag{6}$$

$$\left\{G^{(e)}\right\} = -\int_{V^{(e)}} \left(\lambda \frac{\partial \left[\mathrm{N}^{(e)}\right]^T}{\partial r} \frac{\partial T}{\partial r} + \lambda \frac{\partial \left[\mathrm{N}^{(e)}\right]^T}{\partial z} \frac{\partial T}{\partial z} \right) dV +$$

$$+ \int_{V^{(e)}} \left[\mathrm{N}^{(e)}\right]^T Q dV + \int_{V^{(e)}} \left(\frac{\lambda}{r} \frac{\partial}{\partial r} \left(\left[\mathrm{N}^{(e)}\right]^T r \frac{\partial T}{\partial r} \right) + \lambda \frac{\partial}{\partial z} \left(\left[\mathrm{N}^{(e)}\right]^T \frac{\partial T}{\partial z} \right) \right) dV \tag{7}$$

treti integral lze upravit podle Gaussovy vety na:

$$\int_{S^{(e)}} \left(\frac{\lambda}{r} \left[\mathrm{N}^{(e)}\right]^T r \frac{\partial T}{\partial r} l_r + \lambda \left[\mathrm{N}^{(e)}\right]^T \frac{\partial T}{\partial z} l_z \right) dS \tag{8}$$

$$T = \left[N^{(e)}\right]\left\{T^{(e)}\right\} \tag{9}$$

$$\left\{G^{(e)}\right\} = -\int_{V^{(e)}} \left(\left(\lambda \frac{\partial \left[\mathrm{N}^{(e)}\right]^T}{\partial r} \frac{\partial \left[\mathrm{N}^{(e)}\right]}{\partial r} + \lambda \frac{\partial \left[\mathrm{N}^{(e)}\right]^T}{\partial z} \frac{\partial \left[\mathrm{N}^{(e)}\right]}{\partial z} \right) dV \right) \left\{T^{(e)}\right\} +$$

$$+ \int_{V^{(e)}} \left[\mathrm{N}^{(e)}\right]^T Q dV + \int_{S^{(e)}} \left[\mathrm{N}^{(e)}\right]^T \left(\lambda \frac{\partial T}{\partial r} l_r + \lambda \frac{\partial T}{\partial z} l_z \right) dS \tag{10}$$

$$-\left\{G^{(e)}\right\} = \left(\int_{V^{(e)}} \left[\mathrm{B}^{(e)}\right]^T \left[\mathrm{D}^{(e)}\right]\left[\mathrm{B}^{(e)}\right] dV + \int_{S_2^{(e)}} \alpha \left[\mathrm{N}^{(e)}\right]^T \left[\mathrm{N}^{(e)}\right] dS \right) \left\{G^{(e)}\right\} -$$

$$\int_{V^{(e)}} Q \left[\mathrm{N}^{(e)}\right]^T dV + \int_{S_1^{(e)}} q^{(e)} \left[\mathrm{N}^{(e)}\right]^T dS - \int_{S_2^{(e)}} \alpha T_\infty \left[\mathrm{N}^{(e)}\right]^T dS \tag{11}$$

kde:

$$\left[k^{(e)}\right] = \int_{V^{(e)}} \left[B^{(e)}\right]^T \left[D^{(e)}\right]\left[B^{(e)}\right] dV + \int_{S_2^{(e)}} \alpha \left[N^{(e)}\right]^T \left[N^{(e)}\right] dS \tag{12a}$$

$$\left[f^{(e)}\right] = -\int_{V^{(e)}} Q \left[N^{(e)}\right]^T dV + \int_{S_1^{(e)}} q \left[N^{(e)}\right]^T dS - \int_{S_2^{(e)}} \alpha T_\infty \left[N^{(e)}\right]^T dS \tag{12b}$$

$$[K]\{T\} = \{F\} \tag{13}$$

kde

$$[K] = \sum_{e=1}^{n} \left[k^{(e)}\right] \tag{14a}$$

$$\{F\} = -\sum_{e=1}^{n} \left\{f^{(e)}\right\} \tag{14b}$$

Vypis koeficientu matic pro nasledne programovani

Podle rovnice (21) se ctvercova matice na prave strane sklada ze dvou integralu.
Prvni integral:

$$\int_{V^{(e)}} \left[B^{(e)}\right]^T \left[D^{(e)}\right]\left[B^{(e)}\right] dV = \begin{bmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_k & c_k \end{bmatrix} 2\pi \int_P r dS = \qquad (15)$$

$$= 2\pi \bar{r} P \left[B^{(e)}\right]^T \left[D^{(e)}\right]\left[B^{(e)}\right]$$

kde $\bar{r}$ je prumerny polomer

Druhy integral:
napriklad je povrch mezi body [i,j]

$$\int_{S_2^{(e)}} \alpha \left[N^{(e)}\right]^T \left[N^{(e)}\right] dS = \alpha 2\pi \int_{l_{ij}^{(e)}} \begin{bmatrix} N_i N_i & N_i N_j & N_i N_k \\ N_j N_i & N_j N_j & N_j N_k \\ N_k N_i & N_k N_j & N_k N_k \end{bmatrix} r dl = \frac{\pi \alpha l_{ij}}{6} \begin{bmatrix} 3r_i + r_j & r_i + r_j & 0 \\ r_i + r_j & r_i + 3r_j & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(16)

Prava strana rovnice je dana souctem tri integralu (integrujeme opet pri jednotkove tloustce – za pomoci vztahu (27a) a (27b)).

Prvni integral (pri konstantnim na plose):

$$\int_{V^{(e)}} Q \left[N^{(e)}\right]^T dV = 2\pi Q \int_P \begin{bmatrix} rN_i \\ rN_j \\ rN_k \end{bmatrix} dS = \frac{Q\pi P}{6} \begin{bmatrix} 2r_i + r_j + r_k \\ r_i + 2r_j + r_k \\ r_i + r_j + 2r_k \end{bmatrix} \qquad (17)$$

∘

Druhy a treti integral jsou stejneho tvaru:

$$\int_S \left[N^{(e)}\right]^T dS = \frac{\pi l_{ij}}{3} \begin{bmatrix} 2r_i + r_j \\ 2r_j + r_i \\ 0 \end{bmatrix} + \frac{l_{jk}}{3} \begin{bmatrix} 0 \\ 2r_j + r_k \\ 2r_k + r_j \end{bmatrix} + \frac{l_{ki}}{3} \begin{bmatrix} 2r_i + r_k \\ 0 \\ 2r_k + r_i \end{bmatrix} = I_{23}$$

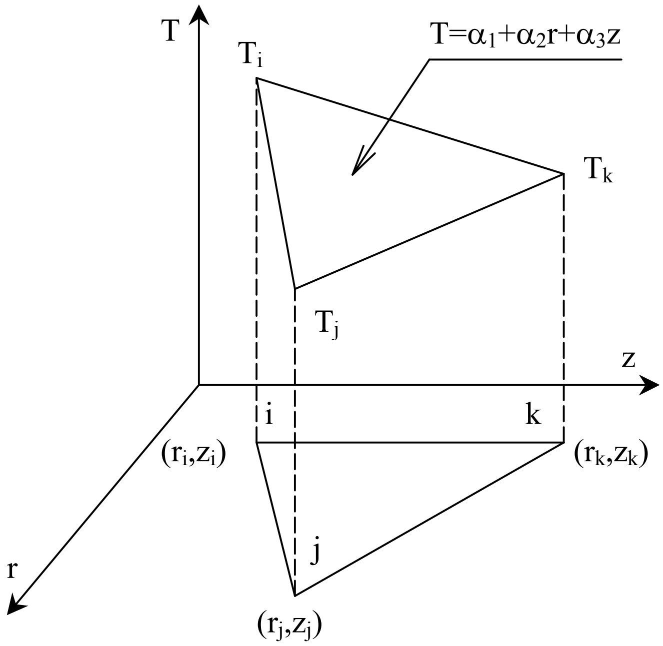
(30)

takze:

$$\int_{S_1^{(e)}} q \left[N^{(e)}\right]^T dS = qI_{23} \qquad (31)$$

$$\int_{S_2^{(e)}} \alpha T_\infty \left[N^{(e)}\right]^T dS = \alpha T_\infty I_{23} \qquad (32)$$

Dodatek - triangulace



Pro teplotu mohu napsat rovnici:

$$T=\alpha_1+\alpha_2 r+\alpha_3 z \qquad (d1)$$

podminky pro tuto rovnici jsou:

$$T=T_i \quad \text{v bode } [r_i,z_i] \qquad (d2)$$
$$T=T_j \quad \text{v bode } [r_j,z_j]$$
$$T=T_k \quad \text{v bode } [r_k,z_k]$$

dosazenim podminek (d2) do rovnice (d1) obdrzime:

$$T_i = \alpha_1+\alpha_2 r_i+\alpha_3 z_i \qquad (d3)$$

$$T_j = \alpha_1+\alpha_2 r_j+\alpha_3 z_j$$
$$T_k = \alpha_1+\alpha_2 r_k+\alpha_3 z_k$$

toto lze psat v maticovem tvaru:

$$\left\{T^{(e)}\right\} = [A]\{\alpha\} \qquad (d5)$$

tedy $$\{\alpha\} = [A]^{-1}\{T^{(e)}\} \tag{d6}$$

$$\alpha_1 = \frac{1}{2P}\left[(r_j z_k - r_k z_j)T_i + (r_k z_i - r_i z_k)T_j + (r_i z_j - r_j z_i)T_k\right] \tag{d7a}$$

$$\alpha_2 = \frac{1}{2P}\left[(z_j - z_k)T_i + (z_k - z_i)T_j + (z_i - z_j)T_k\right] \tag{d7b}$$

$$\alpha_3 = \frac{1}{2P}\left[(r_k - r_j)T_i + (r_i - r_k)T_j + (r_j - r_i)T_k\right] \tag{d7c}$$

kde: P je plocha elementu

$$\begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{vmatrix} = 2P \tag{d8}$$

dosazenim (d7a), (d7b) a (d7c) do rovnice (d1) ziskavam rovnici

$$T = N_i T_i + N_j T_j + N_k T_k \quad \text{nebo} \quad T^{(e)} = [N_i N_j N_k]\begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix} = [N^{(e)}]\{T\} \tag{d9}$$

kde: $$N_i = \frac{1}{2P}[a_i + b_i r + c_i z] \qquad a_i = r_j z_k - r_k z_j \tag{d10a}$$

$$b_i = z_j - z_k$$

$$c_i = r_k - r_j$$

$$N_j = \frac{1}{2P}[a_j + b_j r + c_j z] \qquad a_j = r_k z_i - r_i z_k \tag{d10b}$$

$$b_j = z_k - z_i$$

$$c_j = r_i - r_k$$

$$N_k = \frac{1}{2P}[a_k + b_k r + c_k z] \qquad a_k = r_i z_j - r_j z_i \tag{d10c}$$

$$b_k = z_i - z_j$$

$$c_k = r_j - r_i$$

Casto je treba znat gradient teploty (zapis v maticovem tvaru):

$$\{g\} = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial r} & \frac{\partial N_j}{\partial r} & \frac{\partial N_k}{\partial r} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} \end{bmatrix} \begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix} = \frac{1}{2P} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix} = [B] \begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix}$$

Pouzita literatura:

[1] Larry J. Segerlind, Applied Finite Element Analys, Wiley&Sons, New York,1976

[2] O. C. Zienkiewicz, The Finite Element Method (third edition), McGRAW-HILL Book Company, UK, 1977