FEM – deformacni pole

Celkova potencialni energie elastickeho systemu se rovna energie napjatosti materialu Λ mene prace externich pusobicich na teleso.

$$\Pi=\Lambda-W \tag{1}$$

pro element objemu dV je energie napjatosti dana vztahem:

$$d\Lambda=\frac{1}{2}\{\varepsilon\}^{T}\{\sigma\}-\frac{1}{2}\{\varepsilon_0\}^{T}\{\sigma\} \tag{5}$$

kde $\{\varepsilon\}$ je vektor napjatosti, $\{\varepsilon_0\}$je pocatecni napjatost (nejcasteji vlivem teploty) a $\{\sigma\}$ je vektor napeti. Celkovou energii v elementu vypocitame integraci:

$$\Lambda=\int_V\frac{1}{2}\left(\{\varepsilon\}^{T}\{\sigma\}-\{\varepsilon_0\}^{T}\{\sigma\}\right)dV \tag{6}$$

Protoze je napjatost a napeti svazano elastickymi konstantami, muzeme proto vztah (6) zjednodusit. Napriklad pro 2D problem:

$$\{\varepsilon\}^{T}=\left[\varepsilon_r \quad \varepsilon_z \quad \varepsilon_\theta \quad \gamma_{rz}\right] \tag{7}$$

$$\{\sigma\}^{T}=\left[\sigma_r \quad \sigma_z \quad \sigma_\theta \quad \tau_{rz}\right] \tag{8}$$

a

$$\{\sigma\}=[D]\{\varepsilon\}-[D]\{\varepsilon_0\} \tag{9}$$

kde

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0\\ \nu & 1-\nu & \nu & 0\\ \nu & \nu & 1-\nu & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix} \tag{10}$$

energie systemu je pak:

$$\Lambda=\int_V\frac{1}{2}\left(\{\varepsilon\}^{T}[D]\{\varepsilon\}-2\{\varepsilon\}^{T}[D]\{\varepsilon_0\}+\{\varepsilon_0\}^{T}[D]\{\varepsilon_0\}\right)dV \tag{11}$$

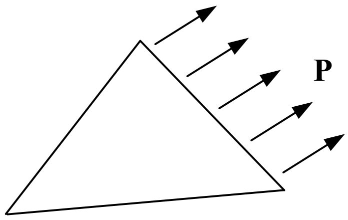
$$\{\varepsilon\}=[B]\{U\} \tag{12}$$

viz dodatek:

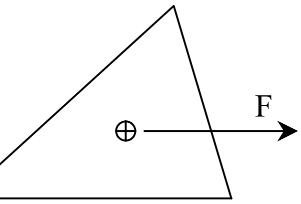
$$\Lambda^{(e)} = \frac{1}{2}\int_{V^{(e)}} \Big(\{U^{(e)}\}^T [B^{(e)}]^T [D^{(e)}][B^{(e)}]\{U^{(e)}\} - \\ - 2\{U^{(e)}\}^T [B^{(e)}]^T [D^{(e)}]\{\varepsilon_0^{(e)}\} + \{\varepsilon_0^{(e)}\}^T [D^{(e)}]\{\varepsilon_0^{(e)}\} \Big) dV \qquad (13)$$

Prace vykonana externimi silami muze byt rozdelena do tri kategorii:
1. vnitrnimi silami
2. prace vykonana tlakovou zatezi
3. silami v jednotlivych uzlech

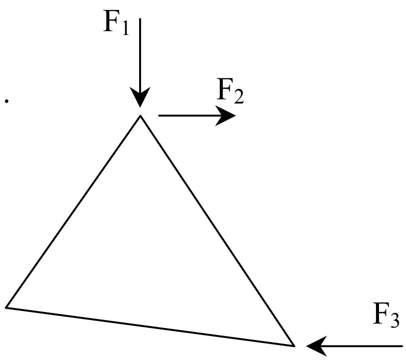
1.



2.



3.



ad 1.

$$W_B^{(e)} = \int_{V^{(e)}} \left(uR^{(e)} + vZ^{(e)}\right) dV = \int_{V^{(e)}} \left\{U^{(e)}\right\}^T \left[N^{(e)}\right]^T \begin{Bmatrix} R^{(e)} \\ Z^{(e)} \end{Bmatrix} dV \tag{14}$$

kde X a Y jsou sily uvnitr elementu

ad 2.

$$W_P^{(e)} = \int_{S^{(e)}} \left(up_r^{(e)} + vp_z^{(e)}\right) dS = \int_{S^{(e)}} \left\{U^{(e)}\right\}^T \left[N^{(e)}\right]^T \begin{Bmatrix} p_r^{(e)} \\ p_z^{(e)} \end{Bmatrix} dS \tag{15}$$

p_x a p_y jsou tlak(tah) na jednotlivy element

ad 3.

$$W_C^{(e)} = \left\{U^{(e)}\right\}^T \left\{P^{(e)}\right\} \tag{16}$$

P vektor sil pusobicich v uzlech elementu

Pak mohu potencialni energii elementu vypocitat podle vztahu:

$$\Pi^{(e)} = \Lambda^{(e)} - W_B^{(e)} - W_P^{(e)} - W_C^{(e)} \tag{17}$$

Potencialni energie cele soustavy je dana vztahem:

$$\Pi = \sum_{e=1}^{n} \Pi^{(e)} \tag{18}$$

Aby byla teleso v rovnovaze, je treba, aby energie byla minimalni

$$\frac{\partial \Pi}{\partial \{U\}} = \{0\} \tag{19}$$

tedy:

$$\{0\} = \sum_{e=1}^{n} \left[\int_{V^{(e)}} \left[B^{(e)}\right]^T \left[D^{(e)}\right] \left[B^{(e)}\right] \left\{U^{(e)}\right\} \right] -$$
$$- \sum_{e=1}^{n} \left[\int_{V^{(e)}} \left[B^{(e)}\right]^T \left[D^{(e)}\right] \left\{\varepsilon_0^{(e)}\right\} dV + \int_{V^{(e)}} \left[N^{(e)}\right]^T \begin{Bmatrix} R^{(e)} \\ Z^{(e)} \end{Bmatrix} dV + \int_{S^{(e)}} \left\{U^{(e)}\right\}^T \left[N^{(e)}\right]^T \begin{Bmatrix} p_r^{(e)} \\ p_z^{(e)} \end{Bmatrix} dS + P^{(e)} \right] \tag{20}$$

nebo:

$$[K]\{U\}=\{F\} \tag{21}$$

kde:

$$[K]=\sum_{e=1}^{n}\left[\int_{V^{(e)}}\left[B^{(e)}\right]^{T}\left[D^{(e)}\right]\left[B^{(e)}\right]\right] \tag{22}$$

a

$$\{F\}=\sum_{e=1}^{n}\left[\int_{V^{(e)}}\left[B^{(e)}\right]^{T}\left[D^{(e)}\right]\left\{\varepsilon_0^{(e)}\right\}dV+\int_{V^{(e)}}\left[N^{(e)}\right]^{T}\begin{Bmatrix}R^{(e)}\\ Z^{(e)}\end{Bmatrix}dV+\int_{S^{(e)}}\left[N^{(e)}\right]^{T}\begin{Bmatrix}p_r^{(e)}\\ p_z^{(e)}\end{Bmatrix}dS+P^{(e)}\right] \tag{23}$$

Vypis koeficientu matic pro nasledne programovani

Leva strava je dana soustavou integralu (22).

$$[K]=\int_V[B]^T[D][B]dV=[\underline{B}]^T[D][\underline{B}]\int_V dV=[\underline{B}]^T[D][\underline{B}]2\pi\underline{r}P \tag{24}$$

prava strana se sklada ze ctyrech casti:
Prvni cast:

$$\int_{V^{(e)}}\left[B^{(e)}\right]^{T}\left[D^{(e)}\right]\left\{\varepsilon_0^{(e)}\right\}dV=\frac{\alpha\Delta TE}{(1-2\nu)}[\underline{B}]^T\begin{Bmatrix}1\\1\\1\\0\end{Bmatrix}2\pi\underline{rP} \tag{25}$$

$$\{\varepsilon_0\}=\alpha\Delta T\begin{Bmatrix}1\\1\\1\\0\end{Bmatrix} \tag{26}$$

Druha cast:

$$\int_V [N]^T \begin{Bmatrix} R \\ Z \end{Bmatrix} dV = \int_A \begin{bmatrix} rN_i & 0 \\ 0 & rN_i \\ rN_j & 0 \\ 0 & rN_j \\ rN_k & 0 \\ 0 & rN_k \end{bmatrix} \begin{Bmatrix} R \\ Z \end{Bmatrix} 2\pi dA = \frac{\pi P}{6} \begin{Bmatrix} (2r_i + r_j + r_k)R \\ (2r_i + r_j + r_k)Z \\ (r_i + 2r_j + r_k)R \\ (r_i + 2r_j + r_k)Z \\ (r_i + r_j + 2r_k)R \\ (r_i + r_j + 2r_k)Z \end{Bmatrix} \tag{27}$$

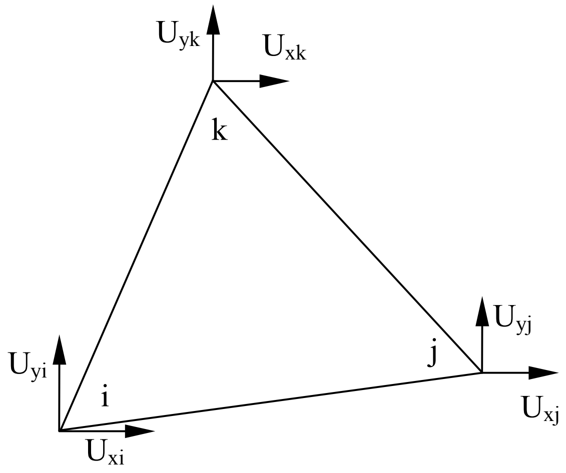
Treti cast:

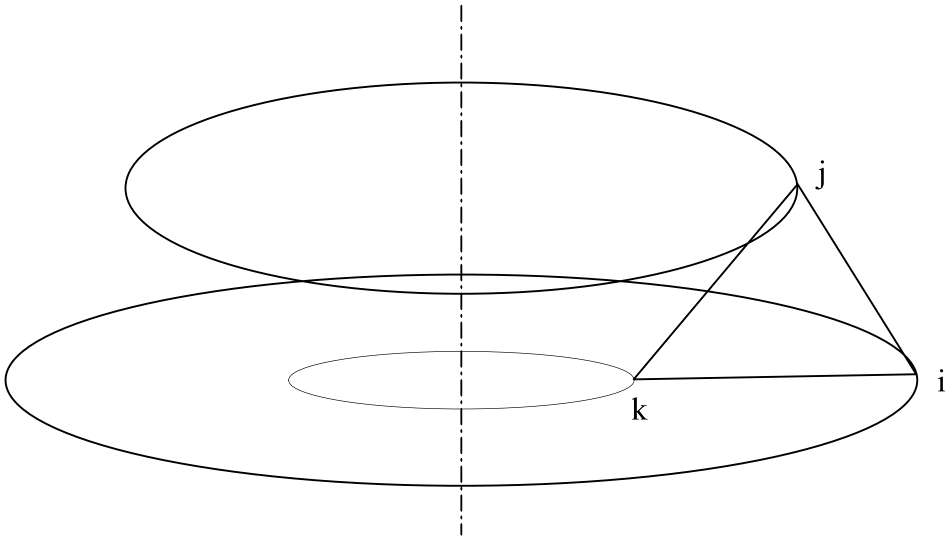
$$\int_{S_{ij}} [N]^T \begin{Bmatrix} p_r \\ p_z \end{Bmatrix} dS = \int_{S_{ij}} \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_k & 0 \\ 0 & N_k \end{bmatrix} \begin{Bmatrix} p_r \\ p_z \end{Bmatrix} dS = \int_{l_{ij}} \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} p_r \\ p_z \end{Bmatrix} 2\pi r dl = \frac{l_{ij}\pi}{3} \begin{Bmatrix} (2r_i + r_j)p_r \\ (2r_i + r_j)p_z \\ (r_i + 2r_j)p_r \\ (r_i + 2r_j)p_z \\ 0 \\ 0 \end{Bmatrix} \tag{28}$$

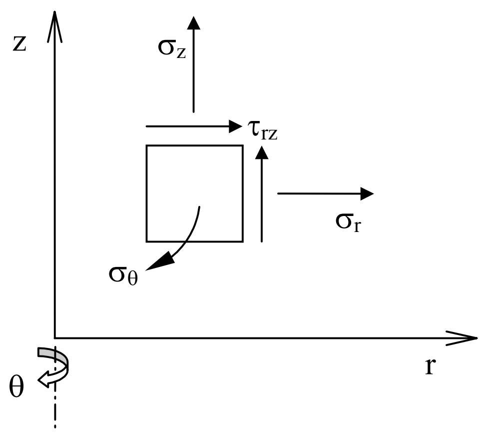
Ctvrta cast:

vektor sil působících v uzlech trolúhelníku.

Dodatek - triangulace







Pro zmenu polohy mohu napsat rovnice:

$$u(x,y)=\alpha_1+\alpha_2 r+\alpha_3 z \quad \text{(d1a)}$$
$$v(x,y)=\alpha_4+\alpha_5 r+\alpha_6 z \quad \text{(d1b)}$$

tuto rovnici mohu prepsat jako:

$$\{\delta\}=\begin{bmatrix}1 & r & z & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & r & z\end{bmatrix}\{\alpha\}=[f]\{\alpha\} \quad \text{(d2)}$$

funkci δ napiseme pro kazdy bod:

$$\{\delta_i\}=\begin{Bmatrix}u_i\\ v_i\end{Bmatrix} \quad \text{(d3a)}$$

$$\{\delta_j\}=\begin{Bmatrix}u_j\\ v_j\end{Bmatrix} \quad \text{(d3b)}$$

$$\{\delta_k\}=\begin{Bmatrix}u_k\\ v_k\end{Bmatrix} \quad \text{(d3c)}$$

pro cely element:

$$\{\delta^{(e)}\}=\begin{Bmatrix}\{\delta_i\}\\ \{\delta_j\}\\ \{\delta_k\}\end{Bmatrix}=\begin{Bmatrix}u_i\\ v_i\\ u_j\\ v_j\\ u_k\\ v_k\end{Bmatrix}=\begin{bmatrix}1 & r_i & z_i & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & r_i & z_i\\ 1 & r_j & z_j & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & r_j & r_j\\ 1 & r_k & z_k & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & r_k & z_k\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\\ \alpha_6\end{Bmatrix} \quad \text{(d4)}$$

nebo:

$$\{\delta^{(e)}\}=[A]\{\alpha\} \quad \text{(d5)}$$

$$\{\alpha\}=[A^{-1}]\{\delta^{(e)}\} \quad \text{(d6)}$$

$$\alpha_1=\frac{1}{2P}\left[(r_j z_k-r_k z_j)u_i+(r_k z_i-r_i z_k)u_j+(r_i z_j-r_j z_i)u_k\right] \quad \text{(d7a)}$$

$$\alpha_2 = \frac{1}{2P}\left[\left(z_j - z_k\right)u_i + \left(z_k - z_i\right)u_j + \left(z_i - z_j\right)u_k\right] \quad \text{(d7b)}$$

$$\alpha_3 = \frac{1}{2P}\left[\left(r_k - r_j\right)u_i + \left(r_i - r_k\right)u_j + \left(r_j - r_i\right)u_k\right] \quad \text{(d7b)}$$

$$\alpha_4 = \frac{1}{2P}\left[\left(r_j z_k - r_k z_j\right)v_i + \left(r_k z_i - r_i z_k\right)v_j + \left(r_i z_j - r_j z_i\right)v_k\right] \quad \text{(d7d)}$$

$$\alpha_5 = \frac{1}{2P}\left[\left(z_j - z_k\right)v_i + \left(z_k - z_i\right)v_j + \left(z_i - z_j\right)v_k\right] \quad \text{(d7e)}$$

$$\alpha_6 = \frac{1}{2P}\left[\left(r_k - r_j\right)v_i + \left(r_i - r_k\right)v_j + \left(r_j - r_i\right)v_k\right] \quad \text{(d7f)}$$

kde: P je plocha elementu

$$\begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{vmatrix} = 2P \quad \text{(d8)}$$

dosazenim (d7a), (d7b) , (d7c), (d7d), (d7e) a (d7f) do rovnice (d2) ziskame rovnici:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \{\delta\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix} = [N]\{U\} \quad \text{(d9)}$$

kde:

$$N_i = \frac{1}{2P}\left[a_i + b_i x + c_i y\right] \qquad a_i = r_j z_k - r_k z_j \quad \text{(d10a)}$$

$$b_i = z_j - z_k$$

$$c_i = r_k - r_j$$

$$N_j = \frac{1}{2P}\left[a_j + b_j x + c_j y\right] \qquad a_j = r_k z_i - r_i z_k \quad \text{(d10b)}$$

$$b_j = z_k - z_i$$

$$c_j = r_i - r_k$$

$$N_k = \frac{1}{2P}\left[a_k + b_k x + c_k y\right] \qquad a_k = r_i z_j - r_j z_i \quad \text{(d10c)}$$

$$b_k = z_i - z_j$$

$$c_k = r_j - r_i$$

casto je znat derivaci:

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_r\\ \varepsilon_z\\ \varepsilon_\theta\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial r}\\ \dfrac{\partial v}{\partial z}\\ \dfrac{u}{r}\\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\end{Bmatrix}= \tag{d11}$$

$$=\frac{1}{2P}\begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0\\ 0 & c_i & 0 & c_j & 0 & c_k\\ \dfrac{a_i}{r}+b_i+c_i\dfrac{z}{r} & 0 & \dfrac{a_j}{r}+b_j+c_j\dfrac{z}{r} & 0 & \dfrac{a_k}{r}+b_k+c_k\dfrac{z}{r} & 0\\ c_i & b_i & c_j & b_j & c_k & b_k\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ u_j\\ v_j\\ u_k\\ v_k\end{Bmatrix}=[B]\{U\}$$

Pouzita literatura:

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[2] O. C. Zienkiewicz, The Finite Element Method (third edition), McGRAW-HILL Book Company, UK, 1977

[3] M. J. Fagan, Finite Element Analysis, Longman Group UK, 1992